UNITED STATES
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THESIS

ON THE STABILITY OF SOLUTIONS OF A
NONLINEAR FIELD EQUATION

by

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ABSTRACT

Solutions to a nonlinear wave equation were analyzed for their stability. The wave equation is a Klein-Gordon equation with the mass replaced by the square of the wave function. This wave equation has propagating solutions which are unbounded or periodic, depending on the sign of the nonlinear term and the propagation speed which can be sub- or super-light velocity. The stability of the periodic sub-light velocity solution was investigated by the method of characteristic exponents and was found to be indifferent. Liapounoff's direct method and Sturrock's analysis of the dispersion relation combined with a WKB technique were applied to a linearized perturbation on a static solution of the field equation. The periodic solution with $\beta^2 < 1$ is stable, while the method of characteristic exponents gives indifference. The super-light velocity solutions are unstable. Due to the limitations of the approximations, it could not be determined whether the instability is absolute or convective.

TABLE OF CONTENTS

INTRODUCTION

In recent years there has been increased interest in nonlinear wave equations in connection with theories of elementary particles. Such equations arise when one attempts to describe a particle or a system of particles, using the theory that the particle itself arises from self-interactions within the wave function. Heisenberg's nonlinear spinor theory, [1], a quantized version of such a theory, was able to reproduce certain features of the spectrum of elementary particles.

Nonlinear differential equations have solutions that are fundamentally different from the solutions of linear equations.[2] They can disappear, become singular, or arise from nothing. A central problem lies in the stability of the solutions.

The purpose of this investigation is to study a possible method of stability analysis as applied to the solution of a wave equation which may serve as representative of a whole class of nonlinear wave problems.

Three methods of stability analysis will be employed. Two of these, Liapounoff's method and the method of characteristic exponents, [3], are interrelated. Liapounoff's method of stability analysis concerns the stability of the equilibrium points. A theorem, also due to Liapounoff, states that the results regarding stability of the linearized theory also hold for the nonlinear case, provided Liapounoff's method is applicable (the characteristic

equation does not have pure imaginary roots). The method of characteristic exponents concerns the stability of periodic solutions.

The third method is the analysis of the dispersion relation as described by P. A. Sturrock [4], which was developed from stability problems in plasma physics. This method is also only applicable in the linearized theory.

The representative differential equation to be used is

$$\nabla^2\Phi - \varepsilon\lambda^2\phi^5 - \frac{1}{c^2}\frac{\partial\phi^2}{\partial t^2} = 0 ,$$

which will be derived in Chapter I from the Klein-Gordon equation.

The particular questions being asked are:

1. What are the static solutions of the equation for given symmetries?
2. Are time dependent solutions to perturbations stable, or for what wave lengths are they unstable?

Chapter I of the paper contains the derivation of the wave equation to be used, an analysis of the types of solutions and some of the pertinent solutions. In the second chapter selected solutions and perturbations are analyzed for stability. Chapter III is a summary of results and conclusions drawn from those results. Two appendixes attached to the paper contain expansions or alternatives to solutions derived in the paper itself.

CHAPTER I

THE NONLINEAR WAVE EQUATION

1.1 DERIVATION OF THE NONLINEAR EQUATION

The wave equation chosen as a starting point for the derivation is the Klein-Gordon wave equation for a free particle:

$$\left(\nabla^2 - \frac{m_0^2 c^2}{h^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \quad . \tag{1}$$

Two reasons may be advanced for this choice. First, the Klein-Gordon equation is the basic equation for relativistic quantum mechanics reflecting relativistic energy conservation. Therefore a nonlinear equation based on this ought to contain similar features which could be interpreted on the basis of the known. Second, both the electromagnetic and Yukawa static potentials can be derived from the relativistic equation by setting the mass term equal to zero or the mass of a pi-meson, respectively. Thus, one can hope to find in the static solution of the nonlinear theory the analogues to the known static potentials.

To derive a nonlinear wave equation, an approach is taken similar to the one that led Heisenberg to his nonlinear spinor equation [1]. Instead of making the mass term a constant, we choose to write $m \sim \phi^2$, and replace the entire term by

$$\varepsilon\lambda^2\phi^4 \quad ,$$

where $\varepsilon = \pm 1$.

This choice appears to be plausible by recalling that in quantum mechanics, the probability of finding a particle

at the position r is $P(r)dr = \phi^*\phi dr$. The particle can be thought of as being smeared out over space and the requirement that

$$\int_V \phi^*\phi \, dV = 1$$

means that the particle remains intact. In a certain sense then, the probability of finding a particle at the position r could be construed to indicate its mass density at the position r. The wave equation to be used is therefore

$$\nabla^2\phi - \varepsilon\lambda^2\phi^5 - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \,. \qquad (2)$$

$\varepsilon = 1$ corresponds to a modified Klein-Gordon equation with the mass replaced by self-interaction of the field. $\varepsilon = -1$ represents the equation of an electromagnetic potential with a source term given by the field self-interaction.

1.2 BEHAVIOR OF THE STATIC SOLUTIONS

The static potential function is the solution of the equation

$$\nabla^2\phi_0 = \varepsilon\lambda^2\phi_0^5 \,. \qquad (3)$$

The equation can be solved using spherical coordinates and assuming that the solution is spherically symmetrical. The equation to be solved is therefore:

$$\frac{d^2\phi_0}{dr^2} + \frac{2}{r}\frac{d\phi_0}{dr} = \varepsilon\lambda^2\phi_0^5 \,.$$

To proceed, assume the solution is of the form

$$\phi_0 = r^n F(u)$$

where u is an unknown function of r. By substitution one obtains:

$$r^n \frac{d^2F}{du^2}(u')^2 + r^n \frac{dF}{du}u'' + 2(n+1)r^{n-1}\frac{dF}{du}u' +$$

$$+ n(n+1)r^{n-2} F = \varepsilon\lambda^2 r^{5n} F^5 \quad .$$

One may choose n and u arbitrarily, and then determine F.

If $u = \ln r$, then the differential equation becomes

$$r^{n-2}\frac{d^2F}{du^2} - r^{n-2}\frac{dF}{du} + 2(n+1)r^{n-2}\frac{dF}{du} + n(n+1)r^{n-2} F = \varepsilon\lambda^2 r^{5n} F^5 .$$

Choose n so that $n - 2 = 5n$; $n = -\frac{1}{2}$. The differential equation then reduces to:

$$\frac{d^2F}{du^2} - \tfrac{1}{4}F - \varepsilon\lambda^2 F^5 = 0 \ .$$

Let

$$z = \frac{dF}{du}$$

$$\frac{dz}{dF} = \frac{1}{z}\frac{d^2F}{du^2}$$

or

$$z\frac{dz}{dF} = \frac{d^2F}{du^2} \quad .$$

Therefore:

$$z\frac{dz}{dF} = \varepsilon\lambda^2 F^5 + \tfrac{1}{4}F.$$

$$\frac{z^2}{2} - \frac{\varepsilon\lambda^2 F^6}{6} + \frac{F^2}{8} = h. \qquad (4)$$

Note that this equation resembles the energy integral of classical mechanics where z is analogous to velocity, the function of F to a negative potential function, in which F plays the role of the space coordinate, and h to the total energy. One

can write the equation in the form:

$$z = \pm\sqrt{2h+\frac{\varepsilon\lambda^2F^6}{3}+\tfrac{1}{4}F^2} \quad . \qquad (5)$$

One can ask what a plot of z versus F (a coordinate-velocity phase diagram in the mechanical analog) would look like for various values of h and ε, provided one requires that z and F remain real. A qualitative analysis yields the results in the following sections.

1.2.1 THE CASE $\varepsilon = 1$

If $\varepsilon = 1$, then

$$z = \pm\sqrt{2h+\frac{\lambda^2F^6}{3}+\tfrac{1}{4}F^2} \quad .$$

If $h = 0$, then for large F

$$z \sim \pm |F^3| \quad .$$

If $h > 0$, z has a minimum of $\sqrt{2h}$ when $F = 0$, but otherwise behaves much in the same manner as for $h = 0$.

If $h < 0$, z can take on all values, and for large F varies as $\pm |F^3|$. In this case, however, when z is zero the absolute value of F is greater than zero.

That is

$$|F\ \min| = \alpha > 0 \quad .$$

A little algebra will yield the one real value of α as

$$\alpha = (A+B)^{\frac{1}{2}} \quad ,$$

$$A = \frac{1}{2\lambda}\left[12b_0^2\lambda + \sqrt{144b_0^4\lambda^2+1}\right]^{\frac{1}{3}}$$

$$B = \frac{1}{2\lambda}\left[12b_0^2\lambda - \sqrt{144b_0^4\lambda^2+1}\right]^{\frac{1}{3}}$$

and $\qquad b_0^2 = |2h|$. The results are shown graphically in Figure 1.

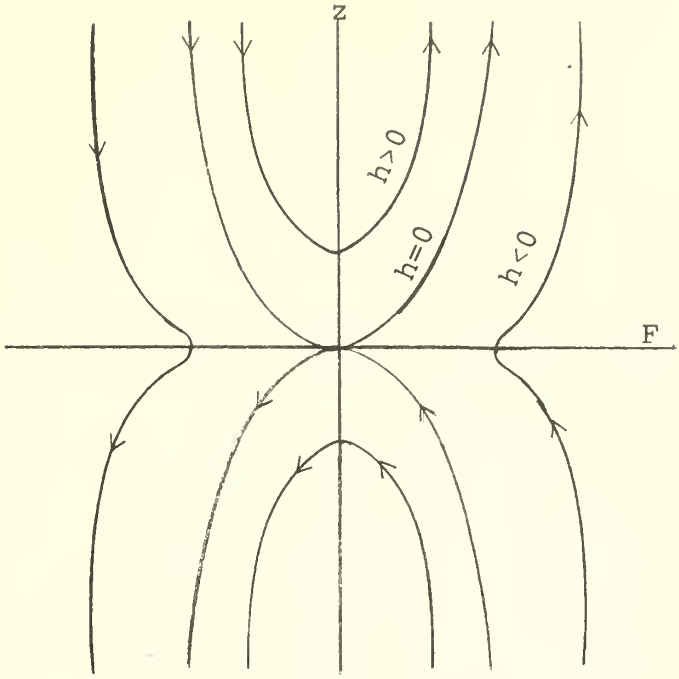


FIGURE 1

Phase Diagram ($\frac{dF}{du}$ vs. F) for Static Case, $\varepsilon = 1$

On each curve the curve parameter is u. The arrows indicate the direction in which the parameter, u, increases. The actual values of the parameter along the curves depend on the particular solution.

1.2.2 THE CASE $\varepsilon = -1$

In this case equation (5) becomes

$$z = \pm\sqrt{2h - \frac{\lambda^2 F^6}{3} + \tfrac{1}{4}F^2} \ .$$

If $h > 0$, a phase plot will yield a closed loop about the origin, where values of z and F outside the loop are forbidden. The characteristics of the loop are dependent on the size of h and λ.

First, one notices that

$$|z| = \sqrt{2h}$$

when

$$F = 0 \ , \ \pm \left(\frac{3}{4\lambda^2}\right)^{\frac{1}{4}}.$$

If z is set to zero, one may solve an equivalent cubic for F max,

$$v^3 - \frac{3v}{4\lambda^2} - \frac{3b_0^2}{\lambda^2} = 0.$$

$$F \max = \pm \ (A+B)^{\frac{1}{2}} \ \text{if} \ 144\lambda^2 b_0^4 \geq 1 \ .$$

$$A = \frac{1}{2\lambda} \ [12\lambda \ b_0^2 + \sqrt{144\lambda^2 b_0^4 - 1}]^{\frac{1}{3}},$$

$$B = \frac{1}{2\lambda} \ [12\lambda b_0^2 - \sqrt{144\lambda^2 b_0^4 - 1}]^{\frac{1}{3}} \ ,$$

and $\qquad b_0^2 = 2h$.

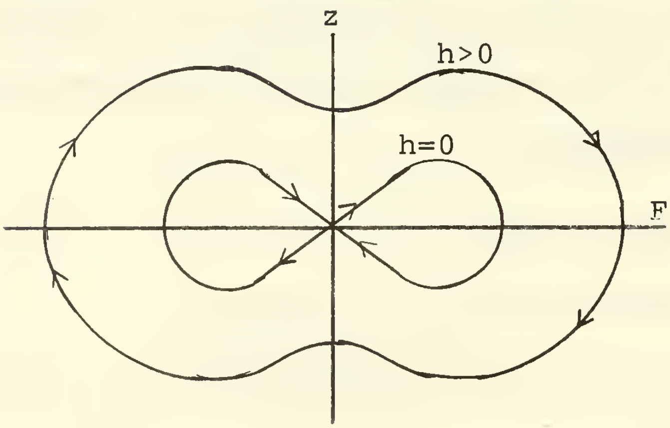


FIGURE 2

Phase Diagram ($\frac{dF}{du}$ vs. F) for Static Case, $\varepsilon = -1$, $h \geq 0$

When $144\lambda^2 b_0^4 < 1$,

$$F \max = \pm \ [\frac{1}{\lambda} \cos \theta/3]^{\frac{1}{2}} \ , \ \cos \theta = 12 b_0^2 \lambda \ .$$

There is only one root for F since if $144\lambda^2 b_0^4 > 1$, two roots of v are imaginary; if $144\lambda^2 b_0^4 \leq 1$, two roots of v are negative and would yield an imaginary F. These results are illustrated in Figure 2.

If h = 0, the double loop in Figure 2 is obtained. This case is only a degenerate case of h > 0. The two nodes at F = 0 meet at the origin. In this case

$$F\ max = \pm \left(\frac{3}{4\lambda^2}\right)^{\frac{1}{4}} .$$

When h < 0, equation (5) may be written

$$z = \pm\sqrt{-b_0^2 - \frac{\lambda^2 F^6}{3} + \tfrac{1}{4}F^2} ,$$

where

$$b_0^2 = |2h| .$$

F now has both a maximum and a minimum. The absolute value of the minima must be greater than zero. The effect is to separate the two lobes for h = 0 in Figure 2 and move them along the F axis. From the solution for the roots of the cubic,

$$v^3 - \frac{3}{4\lambda^2}v + \frac{3b_0^2}{\lambda^2} = 0 ,$$

it can be seen that there is no real F if

$$144\lambda^2 b_0^4 > 1$$

because the roots for v are either negative or imaginary. If

$$144\lambda^2 b_0^4 < 1 ,$$

there will be two solutions for F. These represent the maxima and minima for F.

$$F\ max,\ min = \pm\left(\frac{1}{\lambda}\cos\ \theta/3\right)^{\frac{1}{2}} ,\ \pm\left[\frac{1}{2\lambda}(\sqrt{3}\ \sin\ \theta/3 - \cos\ \theta/3)\right]^{\frac{1}{3}}$$

where $\cos\theta = -12\lambda b_0^2$.

If $144\lambda^2 b_0^4 = 1$, there is one real root for F,

$$F = \pm \left(\frac{1}{2\lambda}\right)^{\frac{1}{2}} .$$

This represents a point solution for F. The results are di-gramed in Figure 3.

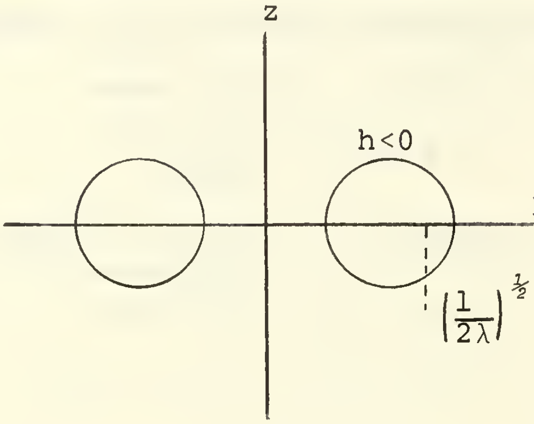


FIGURE 3

Phase Diagram ($\frac{dF}{du}$ vs. F) for Static Case $\varepsilon = -1$, $h < 0$

All the solutions for ϕ_0 will be periodic except when $h = 0$, or when F recedes to a point when $h < 0$.

1.2.3 ANALYTIC SOLUTIONS FOR ϕ_0 WHEN $h = 0$

The two most interesting cases in the last two sections are those for which $h = 0$.

When $\varepsilon = 1$, $h = 0$, one obtains the equation

$$\left(\frac{dF}{du}\right)^2 = \frac{\lambda^2 F^6}{3} + \tfrac{1}{4}F^2 .$$

The equation may be solved by letting

$$F = v^{-\frac{1}{2}} .$$

Then

$$3\left(\frac{dv}{du}\right)^2 = 3v^2 + 4\lambda^2 .$$

$$\int \frac{dv}{(v^2 + \frac{4\lambda^2}{3})^{\frac{1}{2}}} = \ln r/\kappa$$

$$\sinh^{-1} \frac{v\sqrt{3}}{2\lambda} = \ln r/\kappa \ ,$$

which upon transposing the $\sinh^{-1}$ and using the exponential transformation for the sinh, one obtains:

$$v = \frac{\lambda(r^2 - \kappa^2)}{\sqrt{3}\kappa r} ,$$

$$\phi_0 = \left[\frac{\sqrt{3}\kappa}{\lambda(r^2 - \kappa^2)}\right]^{\frac{1}{2}} .$$

This solution, (see Figure 4), represents a potential of infinite height at $r = \kappa$. Suppose this potential represented the potential from a particle. Such a potential would indicate that the particle consisted of a central core of radius κ in which there is no real potential. Outside the core, the potential varies much as a classical $1/r$ potential. The concept of a hard core is not new. Arguments have been advanced that such a core might exist in nuclei between particles. [5]

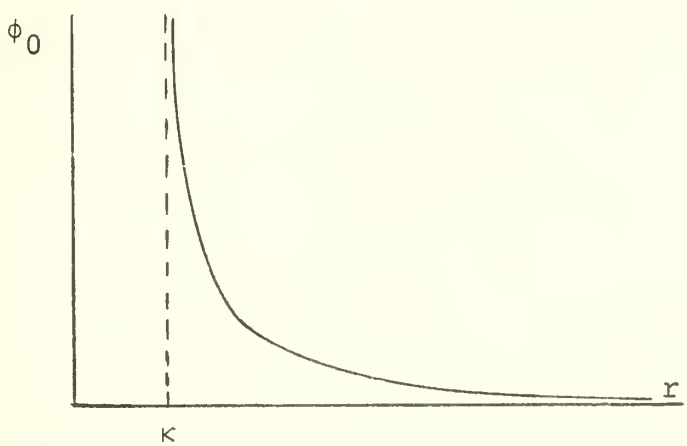


FIGURE 4

Static Solution (ϕ_0 vs. r) $\varepsilon = 1$, $h = 0$, $r > \kappa$

It is surprising that one gets a long range potential instead of the short range Yukawa potential of the Klein-Gordon equation.

Using the solution for ϕ_0 one can determine the applicable regions of the phase diagram in Figure 1.

Since

$$F = v^{-\frac{1}{2}}$$

$$F = \pm \left[\frac{\sqrt{3}\kappa r}{\lambda(r^2-\kappa^2)} \right]^{\frac{1}{2}} .$$

When $F = 0$, $r = \infty$. That is, the saddle point in Figure 1 belongs to $u = \infty$.

When considering the derivatives, one obtains

$$\frac{dF}{du} = z = \frac{1}{r}\frac{dF}{dr} .$$

When $F > 0$, $\frac{dF}{dr} < 0$. Therefore, the applicable region of the phase diagram is the fourth quadrant. When $F < 0$, $\frac{dF}{dr} > 0$. The applicable region in the phase diagram is the second quadrant.

The first and third quadrants must then belong to a different solution. In arriving at the solution, the equation,

$$3(\frac{dv}{du})^2 = 3v^2 + 4\lambda^2 ,$$

was derived and the solution was obtained using the positive root for the derivative. If now the negative root is used, one will eventually get

$$\text{sihn}^{-1} \frac{v\sqrt{3}}{2\lambda} = \ln \kappa/r .$$

The solution for ϕ_0 in this case is

$$\phi_0 = \pm \left[\frac{\sqrt{3}\kappa}{\lambda(\kappa^2-r^2)} \right]^{\frac{1}{2}} .$$

This solution represents a potential well, the general features of which are shown in Figure 5.

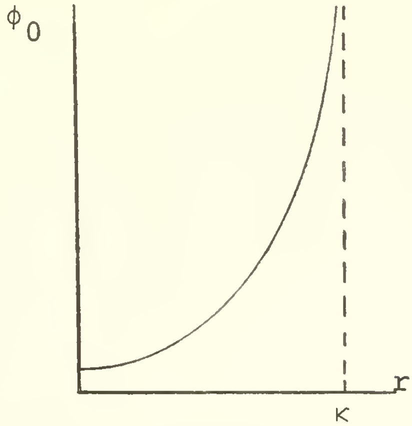


FIGURE 5

Static Solution (ϕ_0 vs. r) $\varepsilon = 1$, $h = 0$, $r < \kappa$

This solution is represented in Figure 1 by the first and third quadrants. The saddle point belongs to $r = 0$, or $u = -\infty$.

If one solves the static equation for $\varepsilon = -1$, $h = 0$, quite a different result occurs.

Let $F = v^{-\frac{1}{2}}$. Then

$$3\left(\frac{dv}{du}\right)^2 = 3v^2 - 4\lambda^2 ,$$

which has the solution

$$\cosh^{-1} \frac{v\sqrt{3}}{2\lambda} = \ln r/\kappa ,$$

where κ is the integration constant. Transposing the inverse cosh, substituting the exponential equivalent for the cosh, and then solving for v, will yield

$$v = \frac{\lambda(r^2+\kappa^2)}{r\kappa\sqrt{3}} .$$

$$\phi_0 = (rv)^{-\frac{1}{2}} = \left[\frac{\kappa\sqrt{3}}{\lambda(r^2+\kappa^2)}\right]^{\frac{1}{2}} . \qquad (7)$$

Equation (7) represents* an equation which bears a striking resemblance to a static electromagnetic field potential. Besides the resemblance to coulomb's law, this solution has another point of similarity with electric potential fields. The differential equation from which it was developed is equation (3) with $\varepsilon = -1$,

$$\nabla^2 \phi_0 = -\lambda^2 \phi_0^5 \tag{8}$$

One might compare this with the static potential equation with a source term,

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \quad ,$$

where ϕ represents potential, ρ the charge density, and ε_0 the permittivity of free space. This potential function, and the wave equation it implies, shall be used for the time dependent stability analysis in the following sections. Other solutions of the static wave equation may be found in Appendix A.

Assume that the ϕ_0 of equation (7) represents the coulomb potential from an electron. One might ask what would the constants κ and λ be, and what would they represent.

To determine the constants two relations will be required. The first is derived from the fact that in electromagnetism

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \; ,$$

and

$$\int_V \nabla^2 \phi dv = -\frac{q}{\varepsilon_0}$$

where q is charge. In this case q = e, the electronic charge.

* Chandrasekhar [6] quotes this solution in the theory of stellar structure.

But for the nonlinear wave equation,

$$\nabla^2\phi_0 = -\lambda^2\phi_0^5 = \frac{1}{\lambda^{\frac{1}{2}}}\left[\frac{\kappa\sqrt{3}}{r^2+\kappa^2}\right]^{\frac{5}{2}} ,$$

and in spherical coordinates

$$\frac{e}{\varepsilon_0} = \int_0^{\pi} d\theta \int_0^{2\pi} d\psi \int_0^{\infty} \frac{1}{\lambda^{\frac{1}{2}}}\, \kappa^{\frac{5}{2}}\, 3^{\frac{5}{4}}\, \frac{r^2\, dr \sin\theta}{(r^2+\kappa^2)^{\frac{5}{2}}} .$$

$$\frac{e}{\varepsilon_0} = \frac{4\pi\cdot 3^{\frac{5}{4}}\,\kappa^{\frac{5}{2}}}{\lambda^{\frac{1}{2}}} \int_0^{\infty} \frac{r^2\, dr}{(r^2+\kappa^2)^{\frac{5}{2}}} .$$

Let $r = \kappa \tan\theta$.

Then the integral becomes

$$\frac{1}{\kappa^2}\int_0^{\pi/2} \sin^2\theta \cos\theta \, d\theta$$

$$\frac{e}{\varepsilon_0} = \frac{4\pi\kappa^{\frac{1}{2}}3^{\frac{1}{4}}}{\lambda^{\frac{1}{2}}} = 9\cdot 10^9 4\pi e$$

and

$$\left(\frac{\kappa}{\lambda}\right)^{\frac{1}{2}} = 3^{\frac{7}{4}}\cdot 10^9 e .$$

Conversations with those performing the colliding beam experiment at Stanford University indicate that coulomb's law holds to within 2% at $r = 3\cdot 10^{-17}$ meters. One may use this data to obtain a second relation between κ and λ.

From the ratio for κ/λ evaluated above,

$$\phi_0 = \frac{9\cdot 10^9 e}{(r^2+\kappa^2)^{\frac{1}{2}}} .$$

Now, let ξ_1 and ξ_2 be defined so that

$$\xi_1 = \frac{1}{r}\ ,\ \xi_2 = \frac{1}{(r^2+\kappa^2)^{\frac{1}{2}}}\ .$$

A plot of these two functions is shown below in Figure 6.

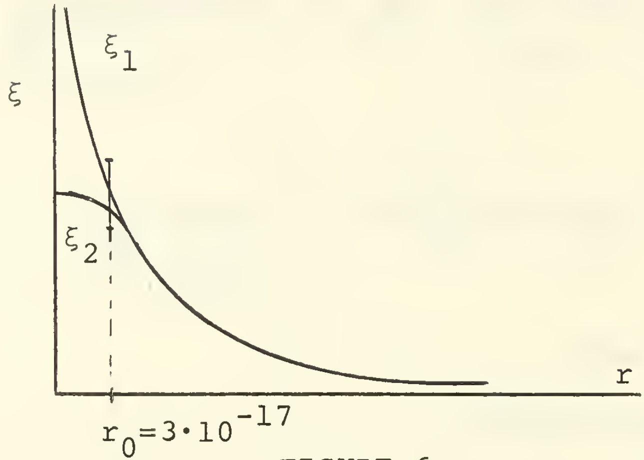


FIGURE 6

Comparison of Coulomb Potential and ϕ_0 For $\varepsilon = -1$, $h = 0$

At r_0, ξ_2 cannot be more than 2% below ξ_1. That is,

$$\xi_1 \geq \xi_2 \geq .98\xi_1 \text{ at } r_0\ .$$

This, in effect, puts an upper bound on κ.

Let

$$\xi_2 = .98\xi_1 \text{ at } r_0.$$

Then:

$$\frac{98}{3}\cdot 10^{17} = \frac{1}{[9\cdot 10^{-34}+\kappa^2]^{\frac{1}{2}}}\ ,$$

$$\kappa \max = .6\ 10^{-17}\ .$$

The range of κ is

$$0 \leq \kappa \leq .6\cdot 10^{-17}$$

using the upper limit of κ to obtain λ, one would get

$$\lambda = 5.$$

1.3 BEHAVIOR OF THE SOLUTIONS OF THE TIME DEPENDENT EQUATION

One can find propagating solutions to the time dependent equation in one dimension. Such a solution will give some idea of the behavior of the solutions for more complicated symmetries. The time dependent wave equation in one dimension is:

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \varepsilon\lambda^2\phi^5 .$$

Let

$$\tau = ct ,$$

$$S = x - \beta\cdot\tau ,$$

$$\beta = \frac{v}{c} .$$

Assume that ϕ is of the form

$$\phi = \phi(S) ,$$

i.e., we confine our attention to propagating solutions.

Then

$$\frac{\partial^2\phi}{\partial x^2} = \frac{d^2\phi}{dS^2} ,$$

and

$$\frac{\partial^2\phi}{\partial\tau^2} = \beta^2\frac{d^2\phi}{dS^2} .$$

The differential equation is

$$\frac{d^2\phi}{dS^2} = \frac{\varepsilon\lambda^2\phi^5}{1-\beta^2} . \tag{9}$$

Let

$$\frac{d\phi}{dS} = \psi .$$

Then

$$\frac{d\psi}{d\phi} = \frac{1}{\psi}\frac{d^2\phi}{dS^2} ,$$

and

$$\psi \frac{d\psi}{d\phi} = \varepsilon \frac{\lambda^2 \phi^5}{1-\beta^2} .$$

As in section 1.2 the first integral yields an equation similar to the energy equation in classical mechanics,

$$\frac{\psi^2}{2} - \frac{\varepsilon\lambda^2\phi^6}{6(1-\beta^2)} = h ,$$

or

$$\psi = \pm\sqrt{2h + \frac{\varepsilon\lambda^2\phi^6}{3(1-\beta^2)}} .$$

A qualitative analysis of this equation using the methods in section 1.2 is presented in the following section.

1.3.1 DISCUSSION OF ψ vs. ϕ

If in the equation for ψ,

$$\varepsilon = 1, \text{ and } \beta^2 < 1 ,$$

curves of ϕ vs. ψ will be the same as those in Figure 1 with F replaced by ϕ and z by ψ. Again this case represents solutions which are unbounded.

If $\varepsilon = 1$, $\beta^2 > 1$, the second term in the equation for ψ is negative and h must be greater than 0. The phase diagram will look something like Figure 7.

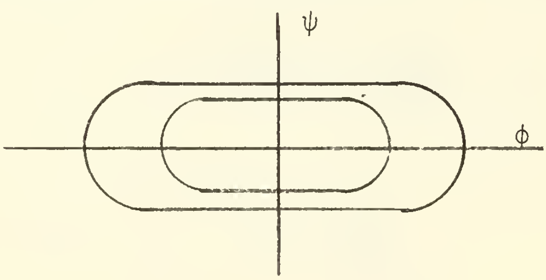


FIGURE 7

Phase Diagram ($\frac{d\phi}{ds}$ vs. ϕ) for Wave Equation With

$\varepsilon = 1$, $\beta^2 > 1$, or $\varepsilon = -1$, $\beta^2 < 1$

One has the result then, that for $\varepsilon = 1$, real sub-light velocity solutions exist which are unbounded. The only periodic solutions are those which exist for super-light velocities.

No other cases need be considered, since if $\varepsilon = -1$, one merely needs to invert β^2 in the last two cases to obtain the same results. That is

$\beta^2 > 1,\ \varepsilon = -1$ corresponds to $\beta^2 < 1,\ \varepsilon = 1$,

and $\beta^2 < 1,\ \varepsilon = -1$ corresponds to $\beta^2 > 1,\ \varepsilon = 1$.

1.3.2 ANALYTIC SOLUTIONS OF THE ONE DIMENSIONAL EQUATION

Because of the redundancy of cases the solutions shall be carried out with $\beta^2 < 1$.

Let

$$2h = \delta b_0^2 \ ; \ \delta = \pm 1 \ .$$

$$\frac{d\phi}{dS} = b_0\sqrt{\delta + \frac{\varepsilon\lambda^2\phi^6}{3(1-\beta^2)b_0^2}} = b_0[\delta + \varepsilon a_0^6\phi^6]^{\frac{1}{2}} \ , \qquad (10)$$

where

$$a_0 = \left(\frac{\lambda^2}{3(1-\beta^2)b_0^2}\right)^{\frac{1}{6}} \ .$$

Let

$$\xi = a_0\phi \ ,$$

then

$$\frac{d\xi}{dS} = a_0 b_0\sqrt{\delta + \varepsilon\xi^6} \ .$$

Let

$$\xi = v^{\frac{1}{2}}$$

then

$$\int \frac{dv}{[v(\delta + \varepsilon v^3)]^{\frac{1}{2}}} = \tfrac{1}{2} a_0 b_0 (S - S_0)$$

where the limits on the integral have been left open since they depend on ε.

If $\varepsilon = \delta = 1$,

the solution is

$$\cos\,\theta = \mathrm{cn}(\gamma,k) \ ,$$

where cn is an elliptic function with modulus k. The variables are

$$\cos\,\theta = \frac{1+(1-\sqrt{3})v}{1+(1+\sqrt{3})v} \ ,$$

$$k^2 = \frac{2+\sqrt{3}}{4} \ ,$$

$$\gamma = 2a_0 b_0 3^{\frac{1}{4}}(S-S_0) \ .$$

Substituting for $\cos\,\theta$ in the solution, one obtains v as

$$v = \frac{1-\mathrm{cn}\ \gamma}{(\sqrt{3}-1)+(\sqrt{3}+1)\,\mathrm{cn}\ \gamma} \ ,$$

and

$$\phi = \frac{1}{a_0}v^{\frac{1}{2}} = \frac{1}{a_0}\left[\frac{1-\mathrm{cn}\ \gamma}{(\sqrt{3}-1)+(\sqrt{3}+1)\,\mathrm{cn}\ \gamma}\right]^{\frac{1}{2}} \ .$$

Let $\delta = -1$, $\varepsilon = 1$.

Then

$$\int_1^v \frac{dv}{[v(v^3-1)]^{\frac{1}{2}}} = 2a_0 b_0(S-S_0) \ .$$

Substitute $v = \frac{1}{w}$.

Then

$$-\int_1^w \frac{dw}{(1-w^4)^{\frac{1}{2}}} = 2a_0 b_0(S-S_0) \ .$$

The solution is

$$w = \text{cn}[2\sqrt{2}a_0 b_0(S-S_0)] = \text{cn}\ \gamma,$$

with modulus

$$k^2 = \tfrac{1}{2}$$

$$\phi = \frac{1}{a_0\sqrt{\text{cn}\ \gamma}} \quad .$$

One other solution may be obtained when b_0^2, or $h = 0$. Then $\frac{\varepsilon}{1-\beta^2}$ must be greater than zero. In this case the equation becomes:

$$\frac{d\phi}{dS} = \frac{\lambda\phi^3}{[3|1-\beta^2|]^{\frac{1}{2}}} = \alpha\phi^3 \quad ,$$

where α can be either positive or negative.

$$\phi = \pm[2\alpha(S_0-S)]^{-\frac{1}{2}} \quad .$$

These solutions are all unbounded. It might be of interest to construct a phase diagram similar to that of Figure 1 with the limiting values of S indicated. The $h = 0$ case is represented by the last solution obtained. When α is positive, and ϕ is negative, the third quadrant curve is obtained. The saddle point belongs to $S = -\infty$ in this case. If α is negative, the second and fourth quadrant curves are applicable. The saddle point belongs to $S = \infty$. The results are shown in Figure 8. The arrows indicate the direction of increasing S.

When $h \neq 0$, the other curves are obtained, in general, using the same methods as in section 1.2.1. If $h > 0$, $\frac{d\phi}{dS}$ is a minimum when $\phi = 0$. This occurs when $\text{cn}\ \gamma = 1$ or $S = S_0$. The derivative approaches infinity when ϕ approaches infinity

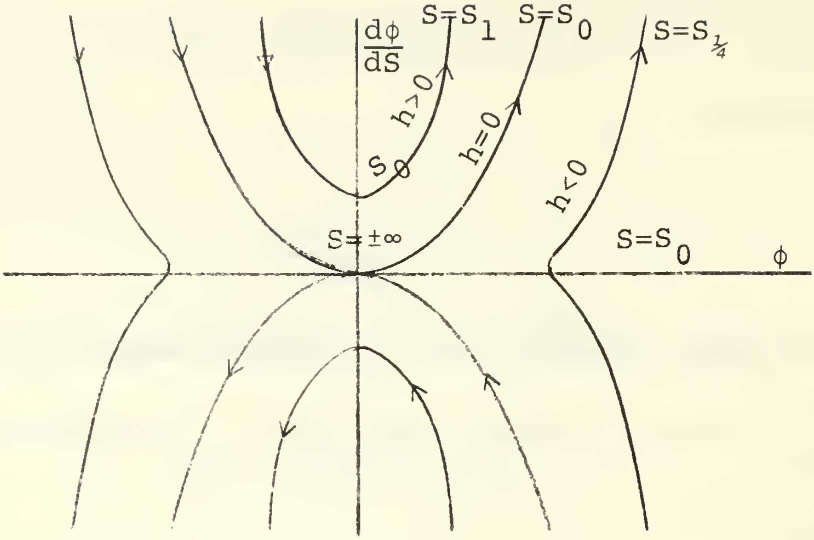


FIGURE 8

Phase Diagram ($\frac{d\phi}{dS}$ vs. ϕ) for Wave Equation with $\varepsilon = +1$, $\beta^2 < 1$

or when

$$\sqrt{3}-1 = (\sqrt{3}+1)\,\mathrm{cn}\,\gamma \ ,$$

for the solution obtained. The cn γ lies somewhere in its second quarter cycle. That is

$$K < \gamma < 2K \ ,$$

where K is the complete elliptic integral of the first kind. S would have some corresponding value, S_1, as labeled on Figure 8. When $h < 0$, $\frac{d\phi}{dS} = 0$ when $\xi = 1$. This corresponds to cn $\gamma = 1$, or $S = S_0$. The derivative approaches infinity when

$$\mathrm{cn}\,\gamma = 0 \ .$$

This occurs when cn γ is at its quarter cycle, or $S = S_{1/4}$. This is the label used in Figure 8.

These solutions became infinite for certain values of S. Adjacent to these infinities the function is not real. It is therefore not possible to construct at any time, t, a

perturbation which is real for all space. These solutions do not allow a ready physical interpretation.

If $\delta = 1$, $\varepsilon = -1$, the equation becomes

$$\int \frac{dv}{[v(1-v^3)]^{\frac{1}{2}}} = 2a_0 b_0 (S-S_0) \; .$$

The solution is

$$\cos\theta = \mathrm{cn}(2a_0 b_0 3^{\frac{1}{4}}(S-S_0)) = \mathrm{cn}\,\gamma \; ,$$

where the modulus of the elliptic function is

$$k^2 = \frac{2-\sqrt{3}}{4} \; ,$$

and

$$\cos\theta = \frac{1-(1+\sqrt{3})v}{1+(\sqrt{3}-1)v} \; .$$

Upon making the proper substitutions and solving for v, one obtains ϕ;

$$\phi = \frac{1}{a_0}\left[\frac{1-\mathrm{cn}\,\gamma}{1+\sqrt{3}+(\sqrt{3}-1)\,\mathrm{cn}\,\gamma}\right]^{\frac{1}{2}} \; . \tag{11}$$

This solution represents the only type of solution that is oscillatory in nature.

CHAPTER II

STABILITY OF SOLUTIONS TO THE WAVE EQUATION

In this section, the stability of the solutions to the wave equation is investigated. The time dependent periodic solution is investigated using the method of characteristic exponents. The following sections will discuss the stability of wave-like perturbations on the static solution for $\varepsilon = -1$. The methods to be used are the characteristic exponent method, Liapounoff's method and Sturrock's stability analysis. Each of these methods shall be described as they occur.

2.1 THE UNBOUNDED TIME DEPENDENT SOLUTIONS

The solutions obtained in section 1.3.2 with $\varepsilon = 1$ in equation 10 all yield unbounded functions, not real for all r and t. These solutions, given in section 1.3.2, are all basically unstable and shall not be discussed further.

2.2 STABILITY OF THE PERIODIC SOLUTION

The solution of equation 10 in section 1.3.2 yielded a periodic solution (equation 11) when $\varepsilon = -1$. The solution shall be investigated for stability using the method of characteristic exponents.

Suppose that one has a system such that

$$\frac{du}{dt} = P(u,v) \ ,$$

and

$$\frac{dv}{dt} = Q(u,v) \ .$$

Suppose the solutions to these equations are known to be

$$u_1 = u_1(t) \text{ and } v_1 = v_1(t).$$

Since these solutions are assumed to be periodic, they must have a period which shall be called T.

Then define

$$h = \frac{1}{T}\int_0^T \left[\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right]dt \ .$$

Then the periodic motion is stable if $h > 0$, and unstable if $h < 0$ [4]. Applying this approach to the specific problem at hand one obtains

$$\frac{d\phi}{dS} = \psi = P \ ,$$

and

$$\frac{d\psi}{dS} = - \frac{\lambda^2 \phi^5}{1-\beta^2} = Q \ .$$

These differential equations are essentially equivalent to derivatives with respect to time if the analysis is carried out for constant position, that is,

$$\frac{d\phi}{dS} = - \frac{1}{\beta}\frac{d\phi}{d\tau}$$

at a given value of x since

$$dS = dx - \beta \, d\tau \ .$$

Carrying through the other operations one finds that

$$\frac{\partial P}{\partial \phi} = \frac{\partial \psi}{\partial \phi} = 0$$

and

$$\frac{\partial Q}{\partial \psi} = \frac{\partial}{\partial \psi}\left(\frac{-\lambda^2 \phi^5}{1-\beta^2}\right) = 0 \ .$$

Therefore $h = 0$.

The theory then predicts that the motion is indifferent to perturbations. That is, the perturbation results in a new amplitude of oscillation which will be maintained.

2.3 STABILITY OF THE PERTURBED STATIC FUNCTION

2.3.1 THE WAVE EQUATION FOR THE PERTURBATION

One can derive a wave equation for a perturbation applied to the static solution. As stated in section 1.2.3, only the case for $\varepsilon = -1$ shall be analyzed so the static potential is represented by equation (7).

The wave equation of interest is

$$\nabla^2\phi + \lambda^2\phi^5 - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \ . \qquad (12)$$

Assume that

$$\phi = \phi_0 + \psi \ ,$$

where ψ is very small.

Since

$$\frac{\partial\phi_0}{\partial t} = 0 \ ,$$

equation (12) becomes

$$\nabla^2\phi_0 + \nabla^2\psi + \lambda^2(\phi_0+\psi)^5 - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0.$$

The third term may be expanded to

$$\lambda^2(\phi_0^5+5\phi_0^4\psi+\ldots+\psi^5) \qquad .$$

But if ψ is very small one may neglect all but the first two terms. Then, since

$$\nabla^2\phi_0 + \lambda^2\phi_0^5 = 0 \ ,$$

one may write

$$\nabla^2\psi + 5\lambda^2\phi_0^4\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0. \qquad (13)$$

Equation (13) shall be used as the wave equation for a perturbation applied to ϕ_0 .

This problem is of interest because, if ϕ_0 represents anything like a particle, then the question arises whether the nonlinear self-interaction allows a stable particle or whether it tends to break up because of application of perturbations.

2.3.2 LIAPOUNOFF'S STABILITY ANALYSIS AND STABILITY OF EQUILIBRIUM

Suppose that one picks a point in space and proceeds to analyze the reaction of a perturbation approaching that point. Then ϕ_0 in equation (13) is a constant. Using the substitutions and the method outlined for the periodic solution in section 2.2, one obtains for a one dimensional perturbation

$$\frac{d^2\psi}{dS^2} = -\frac{5\lambda^2\phi_0^4\psi}{1-\beta^2}\,, \qquad (14)$$

where

$$\frac{5\lambda^2\phi_0^4}{1-\beta^2}$$

is a constant.

Since the equation was derived with x as a constant, the derivative is essentially with respect to time. One can now do a local stability analysis. Where the word local is used to emphasize the fact that only one point in space is being investigated. The equation derived above obviously represents periodic motion if $\beta^2 < 1$, and an unbounded solution if $\beta^2 > 1$.

However, a nonlinear wave equation has been converted to a linear equation by assuming that ψ is small. One might ask

if such a prediction has any validity for the nonlinear problem. Liapounoff's stability analysis will yield the answer to this question.

Let

$$\frac{du}{dt} = P(u,v) \quad ,$$

and

$$\frac{dv}{dt} = Q(u,v) \quad ,$$

and

$$\frac{dv}{du} = \frac{P}{Q} = \frac{cu+dv+Q'(u,v)}{au+bv+P'(u,v)} \quad ,$$

where Q' and P' have no first order and no constant terms. If u and v are near zero, the last equation can be reduced to

$$\frac{dv}{du} = \frac{cu+dv}{au+bv} \quad ,$$

and the equation has been linearized.

Define s so that

$$\begin{vmatrix} a-s & c \\ b & d-s \end{vmatrix} = 0 \ . \qquad (15)$$

Liapounoff's stability criteria are as follows, if the roots s_1 and s_2 of the determinant:

(1) are complex or real, and $Re(s_1, s_2)$ are negative, then the solution is stable.

(2) are complex or real, and at least one of the roots has a positive real part, the solution is unstable.

(3) are imaginary and $Re(s_1, s_2) = 0$, then the stability analysis is questionable. The question as to the solution's stability is not conclusively answered.

Applying this analysis to equation (14) one proceeds as follows:

Let

$$\frac{d\psi}{ds} = \xi ,$$

then

$$\frac{d\xi}{ds} = \frac{-5\lambda^2\phi_0^4\psi}{1-\beta^2} = \alpha\psi ,$$

and

$$\frac{d\xi}{d\psi} = \frac{\alpha\psi}{\xi} .$$

Then in equation (15)

$$c = \alpha,\ b = 1, \text{ and } d = a = 0.$$

Therefore $s^2 - \alpha = 0$, $s = \pm\sqrt{\alpha}$.

If $\beta^2 > 1$, $\alpha > 0$, and the roots of s are real,

$$s = \pm\left[\left|\frac{5\lambda^2\phi_0^4}{1-\beta^2}\right|\right]^{\frac{1}{2}} .$$

Therefore, one can say that if $\beta^2 > 1$, the solution, as already stated, is unstable.

If $\beta^2 < 1$, the roots of s are imaginary. Therefore, the analysis does not give reliable information concerning the stability of the motion. However, the motion is periodic and the method of characteristic exponents in section 2.2 may be used.

Applying this method once again

$$\frac{d\psi}{ds} = P(\psi,\xi) = \xi ,$$

$$\frac{d\xi}{ds} = Q = \frac{-5\lambda^2\phi_0^4\psi}{1-\beta^2} .$$

Then

$$\frac{\partial P}{\partial\psi} = \frac{\partial Q}{\partial\xi} = 0 .$$

Therefore

$$h = \frac{1}{T}\int_0^T \left[\frac{\partial P}{\partial \psi} + \frac{\partial Q}{\partial \xi}\right] dt = 0 \ .$$

Since h is zero, one concludes that the motion is indifferent to perturbations. This result does not change if we take into account all higher order terms in the expansion of $(\phi_0 + \psi)^5$, still considering ϕ_0 constant. The question arises at this point as to whether this approximation to "local" stability or instability, by keeping ϕ_0 a constant and not really considering the perturbation as propagating into regions of changing ϕ_0, might invalidate the conclusions. It certainly is not giving information about propagating instabilities, known also as convective instability. In order to investigate this point the method in the next section will allow removal of the restriction that ϕ_0 remain constant. The stability analysis technique to be used is well known in plasma physics.

2.4 STABILITY ANALYSIS WITH THE WKB METHOD

Another general approach to stability has been developed by Sturrock. It requires that the dispersion relation to the wave equation be obtained. For this analysis a three dimensional, spherically symmetric solution shall be obtained for a perturbation on the static potential function.

2.4.1 AN APPROXIMATE SOLUTION TO THE LINEARIZED WAVE EQUATION

In section 2.3.1, the linearized wave equation for a perturbation on the static potential function was obtained:

$$\nabla^2\psi + 5\lambda^2\phi_0^4\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \ .$$

If the static potential function (equation 7) is put in this equation, one obtains

$$\nabla^2\psi + \frac{15\kappa^2\psi}{(r^2+\kappa^2)^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 .$$

It will be assumed that the perturbation, ψ, is spherically symmetric, thus simplifying the mathematics involved, and amounts to considering s-states only.

The equation then becomes

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{15\kappa^2\psi}{(r^2+\kappa^2)^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 .$$

To solve this equation we assume the solution to be of the form

$$\psi = T(t)R(r) ,$$

and a separation constant α^2. Then we may write:

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{Rr}\frac{dR}{dr} + \frac{15\kappa^2}{(r^2+\kappa^2)^2} = \frac{1}{Tc^2}\frac{d^2T}{dt^2} = -\alpha^2$$

$$\frac{d^2T}{dt^2} + T\alpha^2c^2 = 0.$$

$$T = e^{\pm i\omega t}, \quad \omega = \alpha c .$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + R\left[\frac{15\kappa^2}{(r^2+\kappa^2)^2} + \alpha^2\right] = 0 .$$

Let $rR = u$, then

$$\frac{d^2u}{dr^2} + u\left[\frac{15}{(r^2+\kappa^2)^2} + \alpha^2\right] = 0 .$$

Let

$$r = \kappa x, \quad \alpha^2\kappa^2 = \Omega^2 ,$$

then

$$\frac{d^2u}{dx^2} + u\left[\frac{15}{(x^2+1)^2} + \Omega^2\right] = 0 \ .$$

The exact solution to this equation is rather difficult to obtain. Therefore, the WKB approximation is used to get a solution which may be used for stability analysis.

If one has a differential equation,

$$\frac{d^2u}{dx^2} + K^2u = 0 \ ,$$

then the WKB approximation is to use

$$u = K^{-\frac{1}{2}}e^{i\int K dx} \ .$$

For this case,

$$K^2 = \frac{15}{(x^2+1)^2} + \Omega^2 \ ,$$

and

$$K^{-\frac{1}{2}} = \frac{(x^2+1)^{\frac{1}{2}}}{[15+\Omega^2(x^2+1)^2]^{\frac{1}{4}}} \ .$$

The evaluation of the integral in the exponent will yield elliptic functions of all three types. Such a function will be very difficult to analyze for stability. Therefore, this integral will also be approximated. To do this the Ω^2, x plane shall be divided into two regions. One near the origin and one further from the origin. (See Figure 9).

(a) Assume that

$$\frac{15}{\Omega^2(x^2+1)^2} < 1 \ ,$$

i.e., $\Omega^2(x^2+1)^2$ is large.

Then

$$K = \left[\frac{15}{(x^2+1)^2} + \Omega^2\right]^{\frac{1}{2}} = \Omega\left[\frac{15}{\Omega^2(x^2+1)^2} + 1\right]^{\frac{1}{2}}$$

$$K \simeq \Omega\left[1 + \tfrac{1}{2}\frac{15}{\Omega^2(x^2+1)^2}\right]$$

$$\int K dx \simeq \Omega x + \frac{15x}{4\Omega(x^2+1)} + \frac{15}{4\Omega}\tan^{-1}x \ .$$

(b) Now assume that

$$\frac{\Omega^2(x^2+1)^2}{15} < 1 \ ,$$

i.e., $\Omega^2(x^2+1)^2$ is very small.

Then

$$K = \frac{\sqrt{15}}{x^2+1}\left[1 + \Omega^2\frac{(x^2+1)^2}{15}\right]^{\frac{1}{2}} \ .$$

$$\int K dx \simeq \sqrt{15}\tan^{-1}x + \frac{\Omega^2}{2\sqrt{15}}\left(\tfrac{1}{3}x^3+x\right) \ .$$

To summarize:

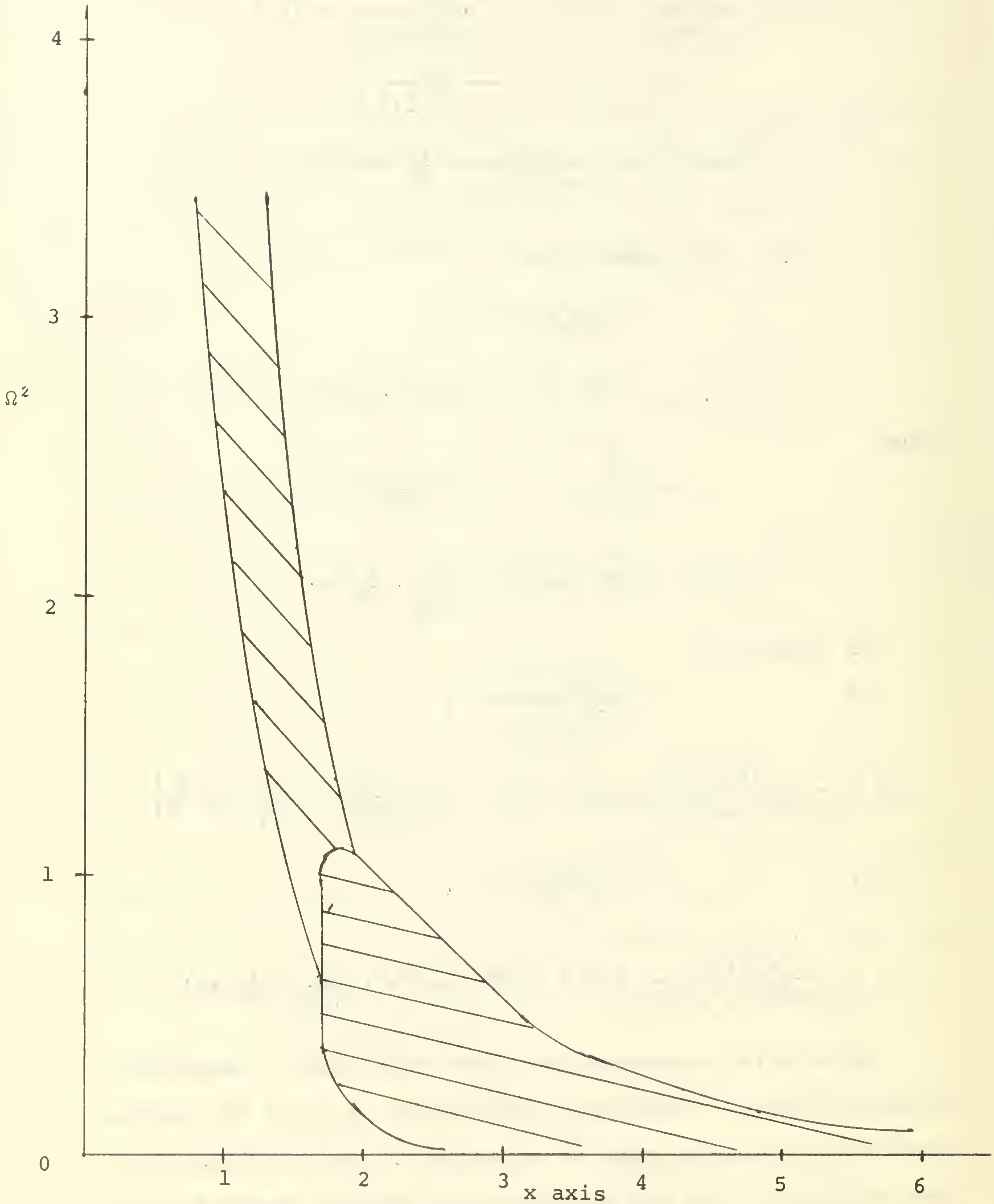
if
$$\frac{15}{\Omega^2(x^2+1)^2} < 1 \ ,$$

$$u = \frac{(x^2+1)^{\frac{1}{2}}}{[15+\Omega^2(x^2+1)^2]^{\frac{1}{4}}} \text{ EXP } i\left[\Omega x + \frac{15x}{4\Omega(x^2+1)} + \frac{15}{4\Omega}\tan^{-1}x\right]$$

if
$$\frac{\Omega^2(x^2+1)^2}{15} < 1 \ ,$$

$$u = \frac{(x^2+1)^{\frac{1}{2}}}{[15+\Omega^2(x^2+1^2]^{\frac{1}{4}}} \text{ EXP } i\left[\sqrt{15}\tan^{-1}x + \frac{\Omega^2}{2\sqrt{15}}\left(\tfrac{1}{3}x^3+x\right)\right] \ .$$

Quite a few approximations have been made in obtaining these solutions. Therefore, the regions in which the approximations are useable must be determined. The condition for the validity of the WKB approximation shall be derived first.



Regions of Validity of WKB Approximations

Figure 9

In the differential equation,

$$\frac{d^2u}{dx^2} + K^2u = 0 \ ,$$

make the independent variable substitution

$$y = \int K dx \quad .$$

Then

$$\frac{d^2u}{dy^2} K^2 + K'\frac{du}{dy} + K^2u = 0 \ .$$

$$\frac{d^2u}{dy^2} + \frac{K'}{K^2}\frac{du}{dy} + u = 0 \ ,$$

where $$K' = \frac{dK}{dx} \ .$$

Now let

$$u = vs, \ s = K^{-\frac{1}{2}}e^{iy} \ .$$

$$\frac{ds}{dy} = \frac{1}{K}\frac{ds}{dx} \ .$$

$$\frac{ds}{dy} = \frac{1}{K}\left(-\frac{1}{2}\frac{K'}{K^{\frac{3}{2}}} + iK^{\frac{1}{2}}\right) e^{iy} \ .$$

$$\frac{d^2s}{dy^2} = \frac{1}{K}\left(\frac{5}{4}\frac{(K')^2}{K^{\frac{7}{2}}} - \frac{1}{2}\frac{K''}{K^{\frac{5}{2}}} - \frac{iK'}{K^{\frac{3}{2}}} - K^{\frac{1}{2}}\right) e^{iy} \ .$$

$$\frac{d^2v}{dy^2} + 2i\frac{dv}{dy} + v\left[\frac{3}{4}\frac{(K')^2}{K^4} - \frac{1}{2}\cdot\frac{K''}{K^3}\right] = 0 \ .$$

If the last term is zero, then the general solution for u is

$$u = K^{-\frac{1}{2}}\left(ae^{-iy} + be^{iy}\right) \ .$$

Therefore the particular solution for u assumed by the WKB approximation is good when the absolute value of the last coefficient is very small.

If ε is defined to be equal to this term, then by differentiating and by algebraic manipulation one obtains for this expression:

$$\varepsilon = \frac{15[15+\Omega^2(x^2+1)^2] - 75\Omega^2x^2(x^2+1)^2}{[15+\Omega^2(x^2+1)^2]^3} \quad .$$

The shaded area near the bottom of Figure 9 is the region of space for which $|\varepsilon| > 0.1$. Within this region the WKB approximation is either not valid or of questionable accuracy.

Next, consider the approximation of $\int K dx$. The approximations used are those for the square root of $(z+1)^{\frac{1}{2}}$.

$$(z+1)^{\frac{1}{2}} = 1 + \tfrac{1}{2}z - \tfrac{1}{4}z^2 \ \ldots \ , \ z < 1 \quad .$$

where only the first two terms were used in the expansion.

The error is not larger than the absolute value of the third term. Again, the value, 0.1, is chosen. For case a,

$$\tfrac{1}{4} z^2 = \frac{15^2}{4\Omega^2(x^2+1)^4} \leq .1 \quad .$$

Making the expression an equality, we can write

$$\Omega^2(x^2+1)^2 = 23.75 \text{ , or roughly ,}$$

$$\Omega^2 = \frac{24}{(x^2+1)^2} \quad .$$

If the same procedure is followed for case b, one obtains

$$\Omega^2 = \frac{9.5}{(x^2+1)^2} \quad .$$

These two functions are plotted in Figure 9. The shaded region between them represents the region where the approximation to the integral is unreliable. The region near the origin is represented by case b; the region above the shaded area by case a.

2.4.2 STURROCK'S STABILITY ANALYSIS

Suppose that the solution of a wave equation yields

$$\phi = e^{i(kx-\omega t)} ,$$

and

$$k = k(\omega) ,$$

$$\omega = \omega(k) .$$

If ω is plotted in the ω_i, ω_r, plane for all real k, then the shape of the plot will yield information about the stability of the wave. Furthermore, a plot of k in the k_i, k_r plane for all real ω will yield information concerning the type of instability of the wave.

For purposes of clarity the possible alternatives shall be divided into four cases.

Case (a) -evanescent waves

These waves are stable. The system is stable if for real k, ω is always real. This situation is diagrammed in Figure 10.

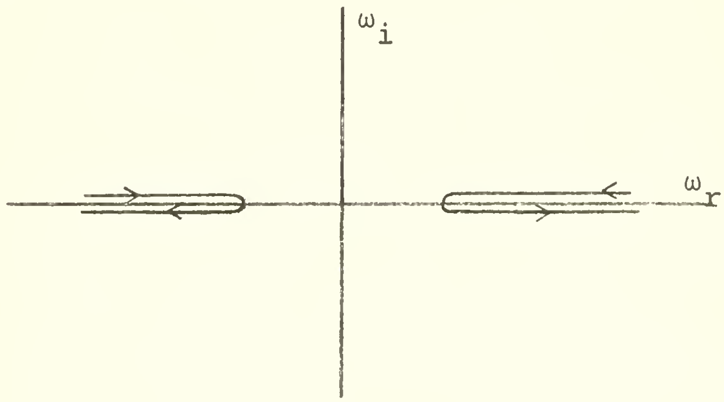


FIGURE 10

Dispersion Relation in the Complex ω Plane for Evanescent Wave

Case (b) - non-convective instability

This case represents waves in which a disturbance grows and expands about the point at which it occurs, but does not travel.

Such a case occurs in the type of plots diagrammed in Figures 11 and 12. Note that $\omega_i/\omega_r \to 0$ as $k \to \pm\infty$, and that k is real for all real ω.

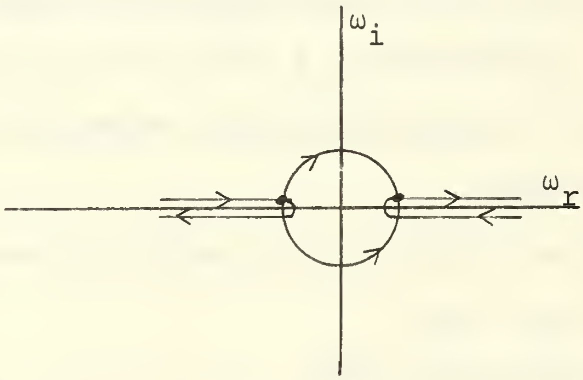


FIGURE 11

Dispersion Relation in the Complex ω Plane

for a Non-convective Instability

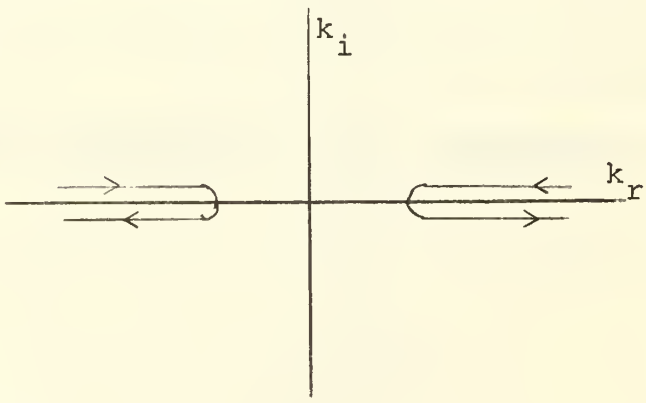


FIGURE 12

Dispersion Relation in the Complex k Plane

for a Non-convective Instability

Case (c) - convective instability

This case represents a wave disturbance which grows and propagates. The ω_i, ω_r plot is the same as Figure 11, but the plot now also has complex values of k for real ω and $k_i/k_r \to 0$ as $\omega \to \pm \infty$. The plot of k_i, k_r would be the same as Figure 11 with the appropriate changes in labeling of axes.

Case (d) In this case $\omega_i/\omega_r \to$ constant $\neq 0$ as $k \to \pm \infty$. This condition is represented in Figure 13.

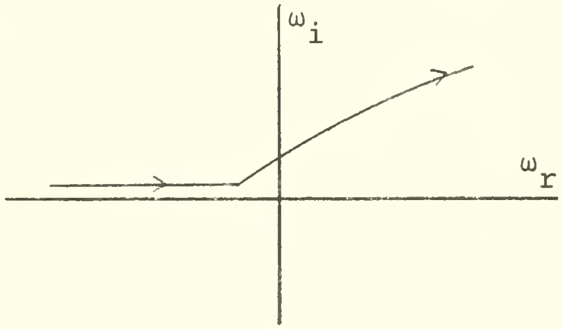


FIGURE 13

Dispersion Relation in the Complex ω Plane for an Undefined Instability

The only conclusion that can be drawn in this case is that the disturbance is unstable. It is Sturrock's contention that this situation will occur only when the model is inexact or incomplete.

To apply Sturrock's theory to the WKB solution obtained for the perturbed wave function, let

$$k = \Omega + \frac{15}{4\Omega(x^2+1)} + \frac{15}{4\Omega x} \tan^{-1} x + \frac{i}{4x} \ln\left[\frac{15}{(x^2+1)^2} + \Omega^2\right] \qquad (16)$$

when
$$\frac{15}{\Omega^2(x^2+1)^2} < 1 \;;$$

and
$$k = \frac{\sqrt{15}}{x}\tan^{-1}x + \frac{\Omega^2}{2\sqrt{15}}\left(\frac{1}{3}x^2+1\right) + \frac{i}{4x}\ln\left[\frac{15}{(x^2+1)^2} + \Omega^2\right] \tag{17}$$

when
$$\frac{\Omega^2(x^2+1)^2}{15} < 1 \quad .$$

Plots of Ω_i vs. Ω_r and k_i vs. k_r are shown in Figures 14 and 15 for $x = 1.5$. The plot of k_i, k_r was obtained directly from equations (16) and (17). The plot of Ω_i, Ω_r was obtained by approximating the logarithms and then solving for Ω.

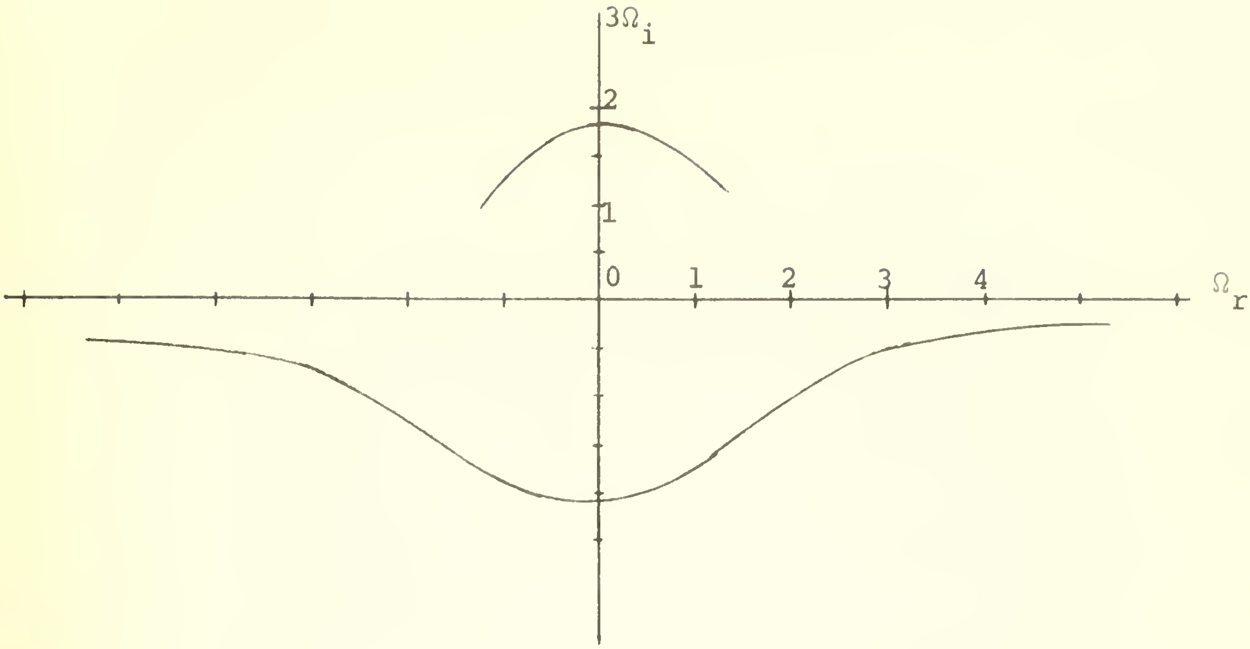
For equation (16) at $x = 1.5$,
$$k = \Omega + \frac{3.61}{\Omega} + i(.166)\ln[1.42+\Omega^2] \quad . \tag{18}$$

The logarithm is approximated by factoring out Ω^2 and dropping the remainder,
$$\ln[1.42+\Omega^2] = 2\ln\Omega + \ln\left[\frac{1.42}{\Omega^2}+1\right] \quad .$$

But from the restrictions on equation (18), due to approximations in the last section, the last term cannot be greater than ln 1.63. Since for this equation $|\Omega| \geq 1.51$, the last term may be dropped if Ω is very large. The logarithm of Ω is then approximated by
$$\ln\Omega = \frac{\Omega-1}{\Omega} \quad .$$

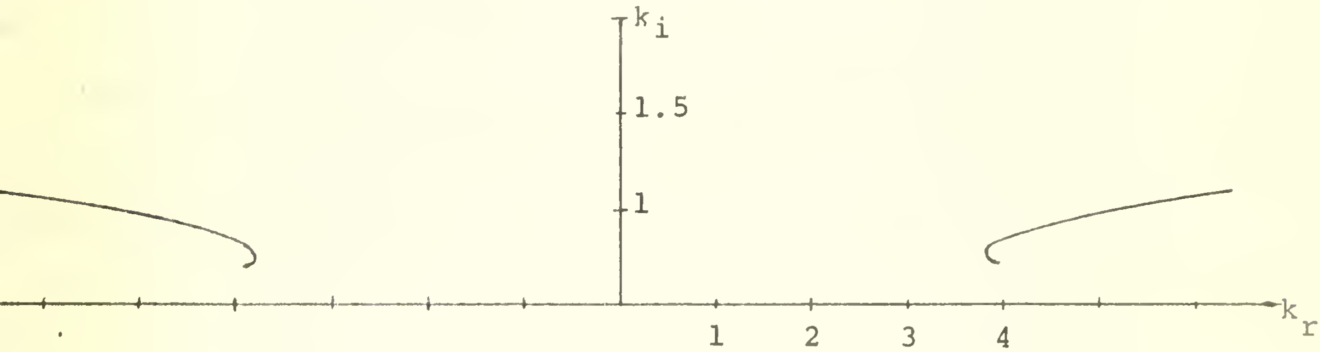
Although this introduces a great deal of error if Ω becomes too large, it does allow one to solve for Ω using only a quadratic and the curve is correct in general features.

The result is
$$k = \Omega + \frac{3.61}{\Omega} + i(.333)\frac{(\Omega-1)}{\Omega} \;,$$



Dispersion Relation in Complex ω Plane for the
Perturbation of the Static Solution

Figure 14



Dispersion Relation in Complex k Plane for the
Perturbation of the Static Solution

Figure 15

which may be solved for Ω,

$$\Omega = \frac{k}{2} - i(.166) \pm \tfrac{1}{2}\sqrt{k^2-14.56+\frac{2}{3}i(2-k)} \quad . \qquad (19)$$

The same general ideas may be carried out for equation (17).

$$k = 2.54 + .266\Omega^2 + i(.117)\Omega^2 + i(.0585) \quad .$$

$$\Omega = \pm\, 2.04e^{-i\ 11.90°}\sqrt{k-2.54-i(.0585)} \qquad (20)$$

This equation is useful only if $|\Omega| < .95$. This restriction severely limits the values allowed by k and the equation is not very useful in obtaining Figure 14. The plot was obtained using equation (19). When the Ω_i, Ω_r plot is displaced below the Ω_r axis, the motion is stable. This is the case for large values of Ω_r in Figure 14. Nearer the origin one can see that the plot looks as though it is unstable. In this region $\frac{\Omega}{k} > 1$. This corresponds to $\beta > 1$ in section 2.3.2 where an unstable solution was predicted. The upper curve in this region has been terminated because neither equation 19 nor 20 yield valid values of Ω. However, the curve seems to curve back towards the origin from calculations with equation (19). This would be consistent with the idea that the solution becomes stable for large values of Ω.

The k_i, k_r plot, Figure 15, does not give as much information. The large gap between the two line segments is due to the limitations on equations (19) and (20).

It can be shown with equation (16) that

$$\lim_{\Omega\to\infty} \frac{k_i}{k_r} = 0.$$

However, due to the limitations on the ranges of equations (16) and (17), one cannot determine very much about the gaps in Figures 14 and 15. Therefore, one cannot say much about the type of instability that exists for small k.

CHAPTER III

SUMMARY AND CONCLUSIONS

3.1 SUMMARY

The nonlinear wave equation,

$$\nabla^2\phi - \varepsilon\lambda^2\phi^5 - \frac{1}{c^2}\,\frac{\partial\phi^2}{\partial t^2} = 0 ,$$

derived from the Klein-Gordon wave equation has been investigated. The $\varepsilon = +1$, this equation represents a true analog to the Klein-Gordon wave equation with the mass replaced by field self-interaction. For $\varepsilon = -1$, the nonlinear term may be interpreted as the source term in the electromagnetic potential equation with the source given by field self-interaction.

Solutions to the static wave equation are characterized by a fundamental length κ and an "energy parameter", h, as integration constants. The term "energy parameter" is used only in the mechanical analog sense. It is found that for $\varepsilon = +1$, the solutions all are unbounded and not everywhere real for all values of h. The case $h = 0$ may behave either as a $1/r$ potential, infinite at the fundamental distance κ, and with a nonreal solution for $r < \kappa$; or as a potential well, centered at the origin, and infinite at $r = \kappa$ with no real solution for $r > \kappa$. The first result may allow the interpretation of a particle with a "hard core". The second result could be interpreted as the basis of a particle whose characteristics are determined by eigenfunctions within the potential well.

For $\varepsilon = -1$, the solutions are everywhere real. In particular, the solution with $h = 0$ is also everywhere finite and has a $1/r$ dependence for $r > \kappa$. This solution may represent an electromagnetic model of a charge with κ as the extension of the charge.

Solutions of the full wave equation were obtained. The solutions are distinguished in their character by their velocity relative to light and by the sign of ε. Those solutions for which $\beta < 1$, $\varepsilon = 1$ or $\beta > 1$, $\varepsilon = -1$, are unbounded and not everywhere real. Those solutions for which $\beta > 1$, $\varepsilon = 1$, or $\beta < 1$, $\varepsilon = -1$ are periodic.

The study of the stability of the solutions with $\varepsilon = -1$ represented the major objective of this paper. The stability was tested in three different ways. The stability of the periodic solutions was tested for one dimensional solutions using the method of characteristic exponents used in nonlinear mechanics. These solutions were found to be indifferent to perturbations, that is the perturbation results in a new solution which will be maintained without blowing up or returning to the original.

Next, the stability of a perturbation on the static potential for $h = 0$ was investigated. The wave equation for the perturbation was linearized and analyzed for stability assuming that ϕ_0 is constant. This amounts to studying "local stability". If $\beta^2 < 1$, Liapounoff's stability analysis gives no information about the stability of equilibrium. The perturbation in this case is periodic and may be analyzed for

stability of the focal point of the phase diagram using the method of characteristic exponents. The result found was again indifferent to stability. If $\beta^2 > 1$, Liapounoff's method indicates instability.

Finally, the linearized perturbation equation was investigated using Sturrock's analysis of the dispersion relation obtained from a WKB approximation to the solution. The dispersion relation obtained was investigated at $r/\kappa = 1.5$, i.e., near the center of the charge. Of the two possible complex roots of ω for real wave number, k, one was found to have only negative imaginary parts indicating a stable solution. The other solution has positive imaginary values around $\omega_r = 0$, indicating instability when $\beta > 1$. The character of the instability could not be determined because of restrictions on the equations used due to the WKB and other approximations.

The results obtained from Sturrock's analysis agrees with the results obtained from the first two methods, if $\beta > 1$. For $\beta < 1$, Sturrock's method predicts stability, whereas the method of characteristic exponents predicts indifference.

3.2 CONCLUSIONS

It has been demonstrated in the paper that a relatively simple nonlinear field equation can serve as a convenient test ground for investigation of stability concerning solutions of nonlinear field equations which might represent particles or electromagnetic sources arising from field self-interaction.

The analysis in this case has shown that certain perturbations are indifferent or stable. The result seems to be sensitive to the change from consideration of "local stability" to consideration of the varying ϕ_0. This suggests that a nonlinear stability analysis might eventually change the result even further. To determine whether or not this is true will require a better approximation to the solution for the perturbation. This may be accomplished by including nonlinear terms in the linearized equation and by carrying the WKB method to high orders of approximation by some kind of iteration. A second alternative would be to seek new avenues of stability analysis, eventually using computer techniques.

It has been shown that there exist other classes of perturbations which are unbounded. It remains to be established whether or not these perturbations are everywhere real and what their physical interpretation would be, perhaps connected with shock-like phenomenon. Answers to these questions again lie in improvement of the WKB approximation and the analysis of the dispersion relation.

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APPENDIX A

The General Solution of the Static Wave Equation

In the main body of the paper potential functions were obtained for the case h = 0 in the equation

$$\left(\frac{dF}{du}\right)^2 = 2h + \frac{\varepsilon\lambda^2 F^6}{3} + \tfrac{1}{4}F^2 \ . \qquad (4)$$

In general, the solutions of the equation are elliptic functions whose form is dependent upon the sign of h and ε, and upon the relative size of h and λ.

In equation (4), let $F = v^{\frac{1}{2}}$ and

$$24h = \delta a_0^2$$

where

$$\delta = \pm 1 \ .$$

The equation then becomes

$$3\left(\frac{dv}{du}\right)^2 = v(\varepsilon 4\lambda^2 v^3 + 3v + \delta a_0^2) \ .$$

This equation, when it yields elliptic functions, may be evaluated using a handbook of elliptic functions. [6] The cubic on the right must first be factored.

$$\frac{3}{4\lambda^2}\ \frac{dv}{du}^2 =$$

$$\varepsilon v\{v-(A+B)][v + \frac{A+B}{2} - \frac{(A-B)}{2}\sqrt{-3}][v + \frac{A+B}{2} + \frac{(A-B)}{2}\sqrt{-3}]\} \qquad (21)$$

where

$$A = \frac{1}{2\lambda}\left[-\frac{\delta a_0^2\lambda}{\varepsilon} + \sqrt{a_0^4\lambda^2+\varepsilon}\right]^{\frac{1}{3}} \qquad (22)$$

$$B = \frac{1}{2\lambda}\left[-\frac{\delta a_0^2\lambda}{\varepsilon} - \sqrt{a_0^4\lambda^2+\varepsilon}\right]^{\frac{1}{3}} \ . \qquad (23)$$

To solve the equation, the solution shall be divided into two cases.

Case 1. $\varepsilon = 1$

If $\delta = 1$ (i.e., $h > 0$) then F can have all values and

$$0 \le v < \infty .$$

The differential equation is

$$\frac{\sqrt{3}}{2\lambda}\int_0^v \frac{dv}{[v(v-R_1)(v-R_2)(v-R_3)]^{\frac{1}{2}}} = \ln\frac{r}{\kappa}$$

where R has been substituted for the roots. For this case, A and B are

$$\frac{1}{2\lambda}\,[-a_0^2\lambda \pm \sqrt{a_0^4\lambda^2+1}]^{\frac{1}{3}} \quad .$$

From an elliptic functions handbook the solution to this equation is obtained by letting

$$k^2 = \frac{(\alpha+\beta)^2-(a+b)^2}{4\alpha\beta} \quad ,$$

$$\cos\,\theta = \frac{2\lambda(\alpha-\beta)v-(a+b)\alpha}{2\lambda(\alpha+\beta)v-(a+b)\alpha} \; ,$$

$$\alpha = (a^2+ 1 + b^2)^{\frac{1}{2}} \quad ,$$

$$\beta = [3(a^2 - 1 + b^2)]^{\frac{1}{2}} \; ,$$

$$a = 2\lambda A, \; b = 2\lambda B \quad .$$

Then

$$\cos\,\theta = \mathrm{cn}\sqrt{\frac{\alpha\beta}{3}}\,\ln\frac{r}{\kappa} \; ,$$

where the modulus of the elliptic function is k.

Let the argument of the cn be x. Then

$$\phi_0 = \left(\frac{v}{r}\right)^{\frac{1}{2}} = \left\{\frac{1}{2\lambda r}\left[\frac{\alpha(a+b)\,\mathrm{tn}^2x/2\;\;\mathrm{dn}^2x/2}{\alpha\mathrm{tn}^2x/2\;\;\mathrm{dn}^2x/2-\beta}\right]\right\}^{\frac{1}{2}} \quad .$$

This solution actually gives a series of unbounded waves. The complicated function is real for small values of

$$tn^2 x/2 \; dn^2 x/2$$

because a + b is negative.

If $\delta = -1$ (i.e., $h < 0$ the differential equation is

$$\frac{\sqrt{3}}{2\lambda} \int_{R_1}^{v} \frac{dv}{[v(v-R_1)(v-R_2)(v-R_3)]^{\frac{1}{2}}} = \ln \frac{r}{\kappa}$$

where the roots are defined by equations (21), (22), and (23). Specifically A and B are

$$a = 2\lambda A = [a_0^2\lambda + \sqrt{a_0^4\lambda^2+1}]^{\frac{1}{3}}$$

$$b = 2\lambda B = [a_0^2\lambda - \sqrt{a_0^4\lambda^2+1}]$$

and $R_1 = A + B$.

The solution will be very similar to that for $\delta = 1$, except that from section 1.2.1, F has a minimum of

$$F \min = (A+B)^{\frac{1}{2}}$$

or $$v \min = A + B.$$

This will necessitate redefining slightly the elliptic function and its modulus.

$$k^2 = \frac{(\alpha+\beta)^2 - (a+b)^2}{4\alpha\beta}$$

$$\alpha^2 = 3(a^2-1+b^2)$$

$$\beta^2 = a^2 + 1 + b^2$$

$$\cos\theta = \frac{2\lambda(\alpha-\beta)v + (a+b)\beta}{2\lambda(\alpha+\beta)v - (a+b)\beta}$$

$$\cos\theta = \mathrm{cn}\sqrt{\frac{\alpha\beta}{3}}\ \ln\frac{r}{\kappa} = \mathrm{cn}\ x$$

$$\phi_0 = \left[\frac{1}{2\lambda r}\left(\frac{\beta(a+b)}{\beta-\alpha \mathrm{tn}^2 x/2\ \ \mathrm{dn}^2 x/2}\right)\right]^{\frac{1}{2}}$$

This function also represents an unbounded wave. The wave minimum occurs periodically and approaches zero as r increases.

Case 2. $\varepsilon = -1$

If $\delta = 1$, the differential equation will be

$$\frac{3}{4\lambda^2}\left(\frac{dv}{du}\right)^2 = -v[(v-R_1)(v-R_2)(v-R_3)]$$

where R_1, R_2, R_3 are defined by equations (21), (22), and (23). A and B will be

$$a = 2\lambda A = [a_0^2\lambda + \sqrt{a_0^4\lambda^2-1}]^{\frac{1}{3}} \quad ,$$

$$b = 2\lambda B = [a_0^2\lambda - \sqrt{a_0^4\lambda^2-1}]^{\frac{1}{3}} \quad .$$

In this instance there are three different solutions. Assume that

$$a_0^4\lambda^2 > 1 \quad .$$

Then

$$\frac{\sqrt{3}}{2\lambda}\int_0^v \frac{dv}{[v(R_1-v)(v-R_2)(v-R_3)]^{\frac{1}{2}}} = \ln\frac{r}{\kappa} \ ,$$

where the roots will be specified as

$R_1 = A + B$ and

R_2 and R_3 are the complex roots.

From section 1.2.2, V has the range

$$0 \leq v \leq A + B \ .$$

Let

$$k^2 = \frac{(a+b)^2-(\alpha-\beta)^2}{4\alpha\beta} ,$$

$$\cos\theta = \frac{(a+b)\beta-2\lambda v(\beta+\alpha)}{(a+b)\beta+2\lambda v(\alpha-\beta)},$$

$$\alpha^2 = 3(a^2+1+b^2),\ \beta^2 = a^2 - 1 + b^2 .$$

Then

$$\cos\theta = \mathrm{cn}\sqrt{\frac{\alpha\beta}{3}}\ \ln\frac{r}{\kappa} = \mathrm{cn}\ x ,$$

where the elliptic function has the modulus k. From the last equation one may solve for v and ϕ_0,

$$\phi_0 = \left[\frac{1}{2\lambda r}\left(\frac{\beta(a+b)\,\mathrm{tn}^2 x/2\,\mathrm{dn}^2 x/2}{\alpha+\beta\,\mathrm{tn}^2 x/2\,\mathrm{dn}^2 x/2}\right)\right]^{\frac{1}{2}}$$

ϕ_0 oscillates in this case with decreasing amplitude as r increases.

Assume $a_0^4\lambda^2 = 1$. This is a special case in which the solution may be expressed in terms of elementary functions. One has

$$\frac{\sqrt{3}}{2\lambda}\int_0^v \frac{dv}{[v(\frac{1}{\lambda}-v)(v+\frac{1}{2\lambda})^2]^{\frac{1}{2}}} = \ln\frac{r}{\kappa} .$$

Make the substitution

$$v = y - \frac{1}{2\lambda} .$$

Then from a table of integrals:

$$\frac{\sqrt{3}}{2\lambda}\int_{\frac{1}{2}\lambda}^{y} \frac{dy}{y(-\frac{3}{4\lambda^2}+\frac{2y}{\lambda}-y^2)^{\frac{1}{2}}} = \left[\sin^{-1}\left(\frac{4\lambda y-3}{2\lambda y}\right)\right]_{\frac{1}{2}\lambda}^{y} .$$

Upon transposing the $\sin^{-1}$ and some manipulation,

$$\phi_0 = \left[\frac{1}{2\lambda r}\left(\frac{1-\cos\ln r/\kappa}{2+\cos\ln r/\kappa}\right)\right]^{\frac{1}{2}} .$$

Assume $a_0^4\lambda^2 > 1$. A better representation of the roots of the cubic in this case would be

$$R_1 = \frac{1}{\lambda}\cos\phi/3 \quad ,$$

$$R_2 = -\frac{1}{2\lambda}(\cos\phi/3+\sqrt{3}\sin\phi/3) \quad ,$$

$$R_3 = \frac{1}{2\lambda}(-\cos\phi/3+\sqrt{3}\sin\phi/3) \quad ,$$

$$\cos\phi = a_0^2\lambda \ .$$

The range of v as calculated in section 1.2.2 is

$$0 \leq v \leq \frac{1}{\lambda}\cos\phi/3 \ .$$

The differential equation is

$$\frac{\sqrt{3}}{2\lambda}\int_{1/\lambda\cos\phi/3}^{v} \frac{dv}{[v(R_1-v)(v-R_2)(v-R_3)]^{\frac{1}{2}}} = \ln\frac{r}{\kappa} \ .$$

Define: $$k^2 = \frac{2\sin\frac{2\phi}{3}}{\sqrt{3}\cos\frac{2\phi}{3}+\sin\frac{2\phi}{3}} \quad ,$$

$$\sin\theta = \sqrt{\frac{[\cos\phi/3+\sqrt{3}\sin\phi/3][\cos\phi/3-\lambda v]}{\cos\phi/3[2\lambda v+\cos\phi/3+\sqrt{3}\sin\phi/3]}} \ .$$

Then

$$\sin\theta = \mathrm{sn}\ \tfrac{1}{2}\left(\frac{\sqrt{3}\cos 2\phi/3+\sin 2\phi/3}{\sqrt{3}}\right)^{\frac{1}{2}} \ln\frac{\kappa}{r}$$

or $$\sin\theta = \mathrm{sn}\ x \ .$$

From the last equation v and ϕ_0 may be obtained.

$$\phi_0 = \left\{\frac{1}{\lambda r}\left[\frac{\mathrm{cn}^2\lambda\cos\phi/3(\cos\phi/3+\sqrt{3}\sin\phi/3)}{\cos\phi/3(1+2\mathrm{sn}^2\lambda)+\sqrt{3}\sin\phi/3}\right]\right\}^{\frac{1}{2}} \ .$$

Let $\delta = -1$. Then the differential equation is

$$\frac{3}{4\lambda^2}\left[\frac{dv}{du}\right]^2 = -v\left(v - \frac{3v}{4\lambda^2} + \frac{a_0^2}{4\lambda^2}\right) \ .$$

A and B in the roots as defined by equations (21), (22), and (23) are

$$a = 2\lambda A = [-a_0^2\lambda + \sqrt{a_0^4\lambda^2-1}]^{\frac{1}{3}}$$

$$b = 2\lambda B = [-a_0^2\lambda - \sqrt{a_0^4\lambda^2-1}]^{\frac{1}{3}} \quad .$$

It was shown in section 1.2.2 that when $h < 0$, F cannot be real unless, $a_0^4\lambda^2 \leq 1$. Therefore this will be the only case considered. The roots can be better expressed as

$$R_1 = \frac{1}{\lambda}\cos\phi/3 \quad ,$$

$$R_2 = \frac{1}{2\lambda}(-\cos\phi/3-\sqrt{3}\sin\phi/3) \quad ,$$

$$R_3 = \frac{1}{2\lambda}(-\cos\phi/3+\sqrt{3}\sin\phi/3) \quad ,$$

$$\cos\phi = -a_0^2\lambda \quad ,$$

and v has the range

$$R_3 \leq v \leq R_1 \quad .$$

Therefore,

$$\frac{\sqrt{3}}{2\lambda}\int_{R_1}^{v} \frac{dv}{[v(R_1-v)(v-R_2)(v-R_3)]^{\frac{1}{2}}} = \ln\frac{r}{\kappa} \quad .$$

Define:

$$k^2 = \frac{\sqrt{3}\cos 2\phi/3+\sin 2\phi/3}{2\sin 2\phi/3} \quad ,$$

$$\sin\theta = \sqrt{\frac{4\sqrt{3}\sin\phi/3(\cos\phi/3-\lambda v)}{(3\cos\phi/3-\sqrt{3}\sin\phi/3)(2\lambda v+\cos\phi/3+\sqrt{3}\sin\phi/3)}} \quad .$$

Then

$$\sin\theta = \mathrm{sn}\left[\left(\frac{\sin 2\phi/3}{2}\right)^{\frac{1}{2}}\ln\frac{\kappa}{r}\right] = \mathrm{sn}\ x \quad ,$$

where the sn x has modulus k. By squaring both sides and solving for v one may obtain

$$\phi_0 = \left\{\frac{1}{2\lambda r}\left[\frac{\sin 2\phi/3(2-\mathrm{sn}^2x)-\sqrt{3}\mathrm{sn}^2x\cos 2\phi/3}{\sin \phi/3(2-\sin^2x)+\sqrt{3}\mathrm{sn}^2x\cos \phi/3}\right]\right\}^{\frac{1}{2}} .$$

This is an oscillating function. In the limit as $\cos\theta = -1$, the term in the brackets becomes identially 1. Then

$$\phi_0 = \left[\frac{1}{2\lambda r}\right]^{\frac{1}{2}}$$

and F is a constant as predicted in section 1.2.2.

APPENDIX B

Exact Solution of $\int K dx$

The exact solution of this integral involves all three types of elliptic functions. One may obtain the solution in the following manner.

$$Kdx = \frac{[15+\Omega^2(x^2+1)^2]^{\frac{1}{2}}dx}{x^2+1} \quad .$$

Let $z = x^2 + 1$. Then the integral becomes

$$\frac{[15+\Omega^2 z^2]^{\frac{1}{2}}dz}{2z(z-1)^{\frac{1}{2}}} \quad .$$

This may be reduced to

$$\frac{15dz}{2\Omega z(z-1)^{\frac{1}{2}}\left(\frac{15}{\Omega^2} + z^2\right)^{\frac{1}{2}}} + \frac{\Omega z \; dz}{2(z-1)^{\frac{1}{2}}\left(\frac{15}{\Omega^2} + z^2\right)^{\frac{1}{2}}} \quad .$$

Define:

$$\beta = (15+\Omega^2)^{\frac{1}{2}} \quad ,$$

$$k^2 = \frac{\beta - \Omega}{2\beta} \quad ,$$

$$\cos\theta = \frac{\beta + \Omega - \Omega z}{\beta - \Omega + \Omega z} = \frac{\beta - \Omega x^2}{\beta + \Omega x^2} \quad ,$$

$$v = F(\theta,k) \quad ,$$

where $F(\theta,k)$ is the elliptic integral of the first kind. The integrand may be replaced by

$$\frac{15}{2}\left(\frac{\Omega}{\beta}\right)^{\frac{1}{2}} \frac{(1+\mathrm{cn}\; v)dv}{\beta+\Omega+(\Omega-\beta)\mathrm{cn}\; v} + \frac{dv}{2}\left(\frac{\Omega}{\beta}\right)^{\frac{1}{2}} \frac{[\beta+\Omega+(\Omega-\beta)\mathrm{cn}\; v]}{1+\mathrm{cn}\; v}$$

where cn v is an elliptic function of v.

Integrating over v one obtains

$$v\left(\frac{\Omega}{\beta}\right)^{\frac{1}{2}} \frac{15}{\beta-\Omega} - (\beta-\Omega)^{\frac{1}{2}}E(v) + (\beta\Omega)^{\frac{1}{2}} \frac{\mathrm{sn}\ v\mathrm{dn}\ v}{1+\mathrm{cn}\ v}$$

$$- \frac{15}{4(\Omega\beta)^{\frac{1}{2}}} \left(\frac{\beta+\Omega}{\beta-\Omega}\right) \left[\pi(\theta, \frac{-\beta k^4}{\Omega}, k)\right]$$

$$+ \frac{\beta}{\sqrt{2}} \tan^{-1}\left[\frac{15}{4\beta} \quad \frac{2}{\beta\Omega}\right]^{\frac{1}{2}} \mathrm{sd}\ v \quad ,$$

where $E(v)$ is the elliptic integral of the second kind, $\pi(\theta, \frac{-\beta k^4}{\Omega}, k)$ is the elliptic integral of the third kind, and the functions sn v, dn v, sd v, are elliptic functions of v.

The following equalities apply:

$$\mathrm{sn}\ v = \sin\theta \ ,$$

$$\mathrm{cn}\ v = \cos\theta \ ,$$

$$\mathrm{dn}\ v = (1-k^2\sin^2\theta)^{\frac{1}{2}},$$

$$\mathrm{sd}\ v = \frac{\sin\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}} \quad .$$

Using these equalities the elliptic functions may be replaced by functions of x. The results yield:

$$v\left(\frac{\Omega}{\beta}\right)^{\frac{1}{2}} \frac{15}{\beta-\Omega} - (\beta\Omega)^{\frac{1}{2}}E(v) - \frac{15}{4(\Omega\beta)^{\frac{1}{2}}} \left(\frac{\beta+\Omega}{\beta-\Omega}\right)\left[\pi(\theta,-\frac{\beta k^4}{\Omega},k)\right]$$

$$+\frac{\beta}{\sqrt{2}} \tan^{-1}\left[\frac{15\ x}{2^{\frac{1}{2}}\beta[\beta^2+2\Omega^2x^2+\Omega^2x^4]^{\frac{1}{2}}}\right] + \frac{\Omega}{\beta+\Omega x^2} \quad (\beta^2+2\Omega^2x^2+\Omega^2x^4)^{\frac{1}{2}} \quad .$$

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13. ABSTRACT

Solutions to a nonlinear wave equation were analyzed for their stability. The wave equation is a Klein-Gordon equation with the mass replaced by the square of the wave function. This wave equation has propagating solutions which are unbounded or periodic, depending on the sign of the nonlinear term and the propagation speed which can be sub- or super-light velocity. The stability of the periodic sub-light velocity solution was investigated by the method of characteristic exponents and was found to be indifferent. Liapounoff's direct method and Sturrock;s analysis of the dispersion relation combined with a WKB technique were applied to a linearized perturbation on a static solution of the field equation. The periodic solution with $\beta^2 < 1$ is stable, while the method of characteristic exponents gives indifference. The super-light velocity solutions are unstable. Due to the limitations of the approximations, it could not be determined whether the instability is absolute or convective.

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